

Computational determination of the largest lattice polytope diameter

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Abstract

A lattice (d, k) -polytope is the convex hull of a set of points in dimension d whose coordinates are integers between 0 and k . Let $\delta(d, k)$ be the largest diameter over all lattice (d, k) -polytopes. We develop a computational framework to determine $\delta(d, k)$ for small instances. We show that $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$; that is, we verify for $(d, k) = (3, 4)$ and $(3, 5)$ the conjecture whereby $\delta(d, k)$ is at most $\lfloor (k + 1)d/2 \rfloor$ and is achieved, up to translation, by a Minkowski sum of lattice vectors.

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1 Introduction

Finding a good bound on the maximal edge-diameter of a polytope in terms of its dimension and the number of its facets is not only a natural question of discrete geometry, but also historically closely connected with the theory of the simplex method, as the diameter is a lower bound for the number of pivots required in the worst case. Considering bounded polytopes whose vertices are rational-valued, we investigate a similar question where the number of facets is replaced by the grid embedding size.

The convex hull of integer-valued points is called a lattice polytope and, if all the vertices are drawn from $\{0, 1, \dots, k\}^d$, it is referred to as a lattice (d, k) -polytope. Let $\delta(d, k)$ be the largest edge-diameter over all lattice (d, k) -polytopes. Naddef [7] showed in 1989 that $\delta(d, 1) = d$, Kleinschmidt and Onn [6] generalized this result in 1992 showing that $\delta(d, k) \leq kd$. In 2016, Del Pia and Michini [3] strengthened the upper bound to $\delta(d, k) \leq kd - \lceil d/2 \rceil$ for $k \geq 2$, and showed that $\delta(d, 2) = \lfloor 3d/2 \rfloor$. Pursuing Del Pia and Michini's approach, Deza and Pournin [5] showed that $\delta(d, k) \leq kd - \lceil 2d/3 \rceil - (k - 3)$ for $k \geq 3$, and that $\delta(4, 3) = 8$. The determination of $\delta(2, k)$ was investigated independently in the early nineties by Thiele [8], Balog and Bárány [2], and Acketa and Žunić [1]. Deza, Manoussakis, and Onn [4] showed that $\delta(d, k) \geq \lfloor (k + 1)d/2 \rfloor$ for all $k \leq 2d - 1$ and proposed Conjecture 1.1.

Conjecture 1.1 $\delta(d, k) \leq \lfloor (k + 1)d/2 \rfloor$, and $\delta(d, k)$ is achieved, up to translation, by a Minkowski sum of lattice vectors.

In Section 2, we propose a computational framework which drastically reduces the search space for lattice (d, k) -polytopes achieving a large diameter. Applying this framework to $(d, k) = (3, 4)$ and $(3, 5)$, we determine in Section 3 that $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$.

Theorem 1.2 *Conjecture 1.1 holds for $(d, k) = (3, 4)$ and $(3, 5)$; that is, $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$, and both diameters are achieved, up to translation, by a Minkowski sum of lattice vectors*

Note that Conjecture 1.1 holds for all known values of $\delta(d, k)$ given in Table 1, and hypothesizes, in particular, that $\delta(d, 3) = 2d$. The new entries corresponding to $(d, k) = (3, 4)$ and $(3, 5)$ are entered in bold.

$\begin{smallmatrix} & k \\ d & \end{smallmatrix}$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	2	3	4	4	5	6	6	7	8	8
3	3	4	6	7	9					
4	4	6	8							
\vdots	\vdots	\vdots								
d	d	$\lfloor \frac{3d}{2} \rfloor$								

Table 1
The largest possible diameter $\delta(d, k)$ of a lattice (d, k) -polytope

2 Theoretical and Computational Framework

Since $\delta(2, k)$ and $\delta(d, 2)$ are known, we consider in the remainder of the paper that $d \geq 3$ and $k \geq 3$. While the number of lattice (d, k) -polytopes is finite, a brute force search is typically intractable, even for small instances. Theorem 2.1, which recalls conditions established in [5], allows to drastically reduce the search space.

Theorem 2.1 *For $d \geq 3$, let $d(u, v)$ denote the distance between two vertices u and v in the edge-graph of a lattice (d, k) -polytope P such that $d(u, v) = \delta(d, k)$. For $i = 1, \dots, d$, let F_i^0 , respectively F_i^k , denote the intersection of P with the facet of the cube $[0, k]^d$ corresponding to $x_i = 0$, respectively $x_i = k$. Then, $d(u, v) \leq \delta(d-1, k) + k$, and the following conditions are necessary for the inequality to hold with equality:*

- (1) $u + v = (k, k, \dots, k)$,
- (2) any edge of P with u or v as vertex is $\{-1, 0, 1\}$ -valued,
- (3) for $i = 1, \dots, d$, F_i^0 , respectively F_i^k , is a $(d-1)$ -dimensional face of P with diameter $\delta(F_i^0) = \delta(d-1, k)$, respectively $\delta(F_i^k) = \delta(d-1, k)$.

Thus, to show that $\delta(d, k) < \delta(d-1, k) + k$, it is enough to show that there is no lattice (d, k) -polytope admitting a pair of vertices (u, v) such that $d(u, v) = \delta(d, k)$ and the conditions (1), (2), and (3) are satisfied. The computational framework to determine, given (d, k) , whether $\delta(d, k) = \delta(d-1, k) + k$ is outlined below and illustrated for $(d, k) = (3, 4)$ or $(3, 5)$.

Algorithm to determine whether $\delta(d, k) < \delta(d - 1, k) + k$

Step 1: INITIALIZATION

Determine the set \mathcal{F} of all the lattice $(d - 1, k)$ -polytopes P such that $\delta(P) = \delta(d - 1, k)$. For example, for $(d, k) = (3, 4)$, the determination of all the 335 lattice $(2, 4)$ -polygons P such that $\delta(P) = 4$ is straightforward.

Step 2: SYMMETRIES

Consider, up to the symmetries of the cube $[0, k]^d$, the possible entries for a pair of vertices (u, v) such that $u + v = \{k, k, \dots, k\}$. For example, for $(d, k) = (3, 4)$, the following 6 vertices cover all possibilities for u up to symmetry: $(0, 0, 0)$, $(0, 0, 1)$, $(0, 0, 2)$, $(0, 1, 1)$, $(0, 1, 2)$, and $(0, 2, 2)$, where $v = (4, 4, 4) - u$.

Step 3: SHELLING

For each of the possible pairs (u, v) determined during Step 2, consider all possible ways for $2d$ elements of the set \mathcal{F} determined during Step 1 to form the $2d$ facets of P lying on a facet of the cube $[0, k]^d$. For example, for $(d, k) = (3, 4)$ and $u = (0, 0, 0)$, we must find 6 elements of \mathcal{F} , 3 with $(0, 0)$ as a vertex, and 3 with $(4, 4)$ as a vertex. In addition, if an edge of an element of \mathcal{F} with u or v as vertex is not $\{-1, 0, 1\}$ -valued, this element is disregarded.

Note that since the choice of an element of \mathcal{F} defines the vertices of P belonging to a facet of the cube $[0, k]^d$, the choice for the next element of \mathcal{F} to form a shelling is significantly restricted. In addition, if the set of vertices and edges belonging to the current elements of \mathcal{F} considered for a shelling includes a path from u to v of length at most $\delta(d - 1, k) + k - 1$, a shortcut between u and v exists and the last added elements of \mathcal{F} can be disregarded.

Step 4. INNER POINTS

For each choice of $2d$ elements of \mathcal{F} forming a shelling obtained during Step 3, consider the $\{1, 2, \dots, k - 1\}$ -valued points not in the convex hull of the vertices of the $2d$ elements of \mathcal{F} forming a shelling. Each such $\{1, 2, \dots, k - 1\}$ -valued point is considered as a potential vertex of P in a binary tree. If the current set of edges includes a path from u to v of length at most $\delta(d - 1, k) + k - 1$, a shortcut between u and v exists and the corresponding node of the binary tree can be disregarded, and the binary tree is pruned at this node.

A convex hull and diameter computation are performed for each node of the obtained binary tree. If there is a node yielding a diameter of $\delta(d - 1, k) + k$

we can conclude that $\delta(d, k) = \delta(d - 1, k) + k$. Otherwise, we can conclude that $\delta(d, k) < \delta(d - 1, k) + k$. For example, for $(d, k) = (3, 5)$, no choice of 6 elements of \mathcal{F} forming a shelling such that $d(u, v) \geq 10$ exist, and thus Step 4 is not executed.

3 Computational Results

For $(d, k) = (3, 4)$, a shelling exists for which path lengths are not decidable by the algorithm without convex hull computations. However, this shelling only achieves a diameter of 7. For $(d, k) = (3, 5)$ the algorithm stops at Step 3, as there is no combination of 6 elements of \mathcal{F} which form a shelling such that $d(u, v) \geq \delta(2, 5) + 5$. Thus, no convex hull computations are required for $(d, k) = (3, 5)$. A shortcut from u to v is typically found early on in the shelling, which leads to the algorithm terminating quickly. Run on a 2009 Intel[®] Core[™]2 Duo 2.20GHz CPU, the algorithm is able to terminate for $(d, k) = (3, 4)$ and $(3, 5)$ in under a minute. Consequently, $\delta(3, 4) < 8$ and $\delta(3, 5) < 10$. Since the Minkowski sum of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, and $(1, 1, 1)$ forms a lattice $(3, 4)$ -polytope with diameter 7, we conclude that $\delta(3, 4) = 7$. Similarly, since the Minkowski sum of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, $(0, 1, -1)$, $(1, 0, -1)$, and $(1, -1, 0)$ forms, up to translation, a lattice $(3, 5)$ -polytope with diameter 9, we conclude that $\delta(3, 5) = 9$. Computations for additional values of $\delta(d, k)$ are currently underway. In particular, the same algorithm may determine whether $\delta(d, k) = \delta(d - 1, k) + k$ or $\delta(d - 1, k) + k - 1$ for $(d, k) = (5, 3)$ and $(4, 4)$ provided the set of all lattice $(d - 1, k)$ -polytopes achieving $\delta(d - 1, k)$ is determined for $(d, k) = (5, 3)$ and $(4, 4)$. Similarly, the algorithm could be adapted to determine whether $\delta(d, k) < \delta(d - 1, k) + k - 1$ provided the set of all lattice $(d - 1, k)$ -polytopes achieving $\delta(d - 1, k)$ or $\delta(d - 1, k) - 1$ is determined. For example, the adapted algorithm may determine whether $\delta(3, 6) = 10$.

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